An apparent paradox: Catt's anomaly

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An apparent paradox: Catt's anomaly

M Pieraccini and S Selleri

Department of Information Engineering, University of Florence, Via Santa Marta, 3, I-50139 Firenze, Italy

E-mail: massimiliano.pieraccini@unifi.it

Abstract

Catt's anomaly is a sort of 'thought experiment' (a *gedankenexperiment*) where electrons seem to travel at the speed of light. Although its author argued with conviction for many years, it has a clear and satisfactory solution and it can be considered indubitably just an apparent paradox. Nevertheless, it is curious and very intriguing, and able to capture the attention of students.

Introduction

'Catt's anomaly', after Catt [1] who supported it with conviction for many years, is just an apparent paradox about the propagation of electromagnetic waves on a transmission line. Nevertheless, it is didactically effective. The teacher can begin the lesson by capturing the attention of the students with the 'dramatic' story of the conflict between an unconventional man (Catt) and academia [2]. Afterwards, the teacher presents an intriguing (apparent) paradox. Finally, the teacher gives the solution as a sort of twist. This 'narrative structure' could be a valuable way to maintain high attention and interest of students during class.

The subject is understandable at the knowledge level of secondary school students. The most advanced concept involved is Gauss's law.

The anomaly

Let us consider a transmission line, connected to a load and a generator. Let us assume for simplicity a two-wire transmission line in air, but coaxial cables or other lines would behave in the same way. At $t < 0$, the switch between the generator and the line is opened (figure 1(a)). At $t = 0$, the switch closes. A transverse electromagnetic (TEM) wave begins to travel along the line at the speed of light in vacuum, c (figure 1(b)).

With reference to figure 1, this means that there is a travelling electric field **E**, whose lines of force start from the upper conductor and end on the lower conductor. Gauss's law states that the flux of the electric field across a closed surface equals the total charge enclosed within the surface divided by the medium's permittivity, which, in air, is practically that of vacuum, ε_0. Therefore, a small cylindrical surface around one of the conductors exhibits a null flux when it has not been reached by the wave (figure 1(b), on the right), but once reached, the flux is not null anymore due to the presence of the electric field (figure 1(b), on the left) and a charge must be distributed on the wire.

The so-called Catt's anomaly is as follows: where is this charge coming from? Electrons are the moving charges in metal, but they have mass and cannot travel at the speed of light in vacuum. How can they follow the electric field?

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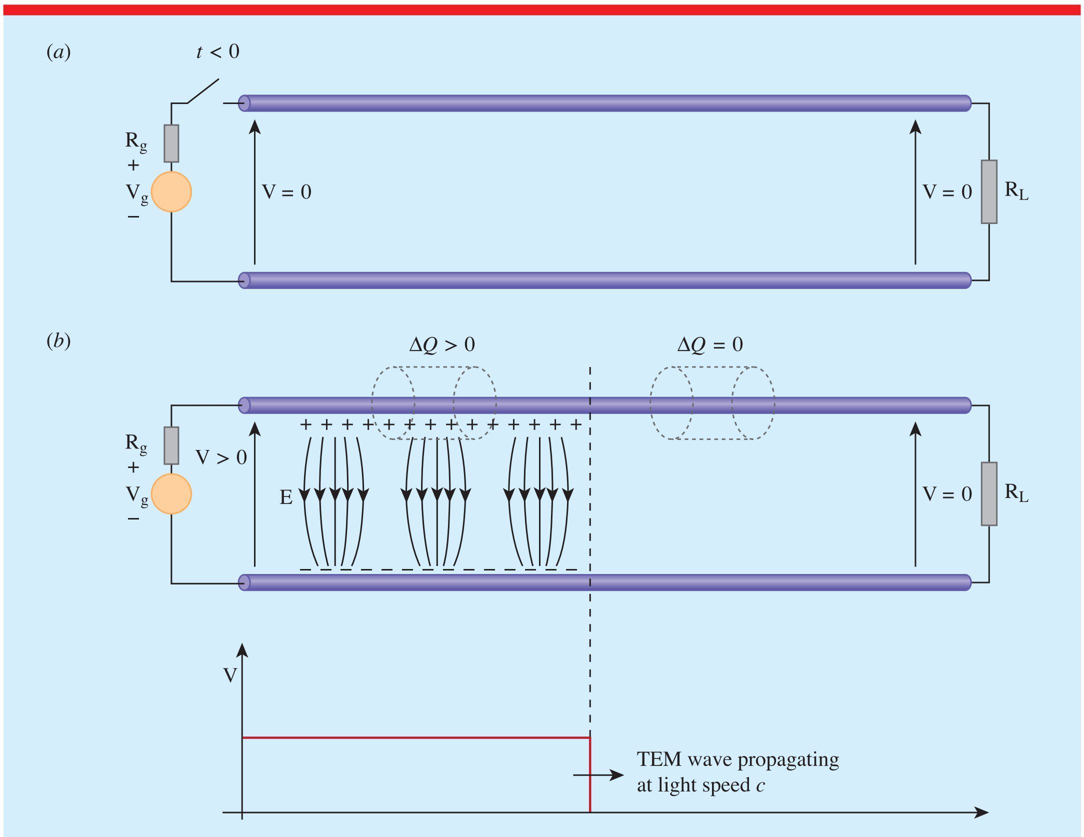

Figure 1. A transmission line connected to a load and a generator via an opened switch (a). The same transmission line after the switch has been closed but before the TEM wave has reached the load (b).

It is important to note that the only knowledge students need is Gauss's law. At this point the teacher should encourage students to propose solutions and ideas. It could be useful also to give a break or postpone the explanation to the next lesson.

The solution

The key idea of the explanation of this apparent paradox is related to the great number of electrons in metal. Although each single electron is not able to travel at the speed of light, a great number of slow electrons are able to produce a current as fast as an electromagnetic wave travelling at the speed of light in the conductor.

A possible analogy is the start of a marathon: the referee shoots the starting gun, the sound of the bang propagates in air, and each athlete begins to run when they hear it. The apparent effect is a wave of running athletes that propagates along the street at the speed of sound, even if obviously no one person can run so fast.

Let us make this concept rigorous. In figure 2 two thin wires of radius a are sketched and the wave is shown as it travels through a sampling cylindrical volume of height Δx.

The wave travels at the speed of light, c, from point x to point $x + \Delta x$ in the time interval $\Delta t = \Delta x/c$ (figure 2(c)). During this time a current I flows in the sampling volume from its left side at x, equalling

$$I = \pi a^2 v q N, \qquad (1)$$

where v is the drift velocity of the charges (in practice electrons, and the speed is much lower than the speed of light), q is the elementary charge (1.602×10^{-19} C) and N is the concentration

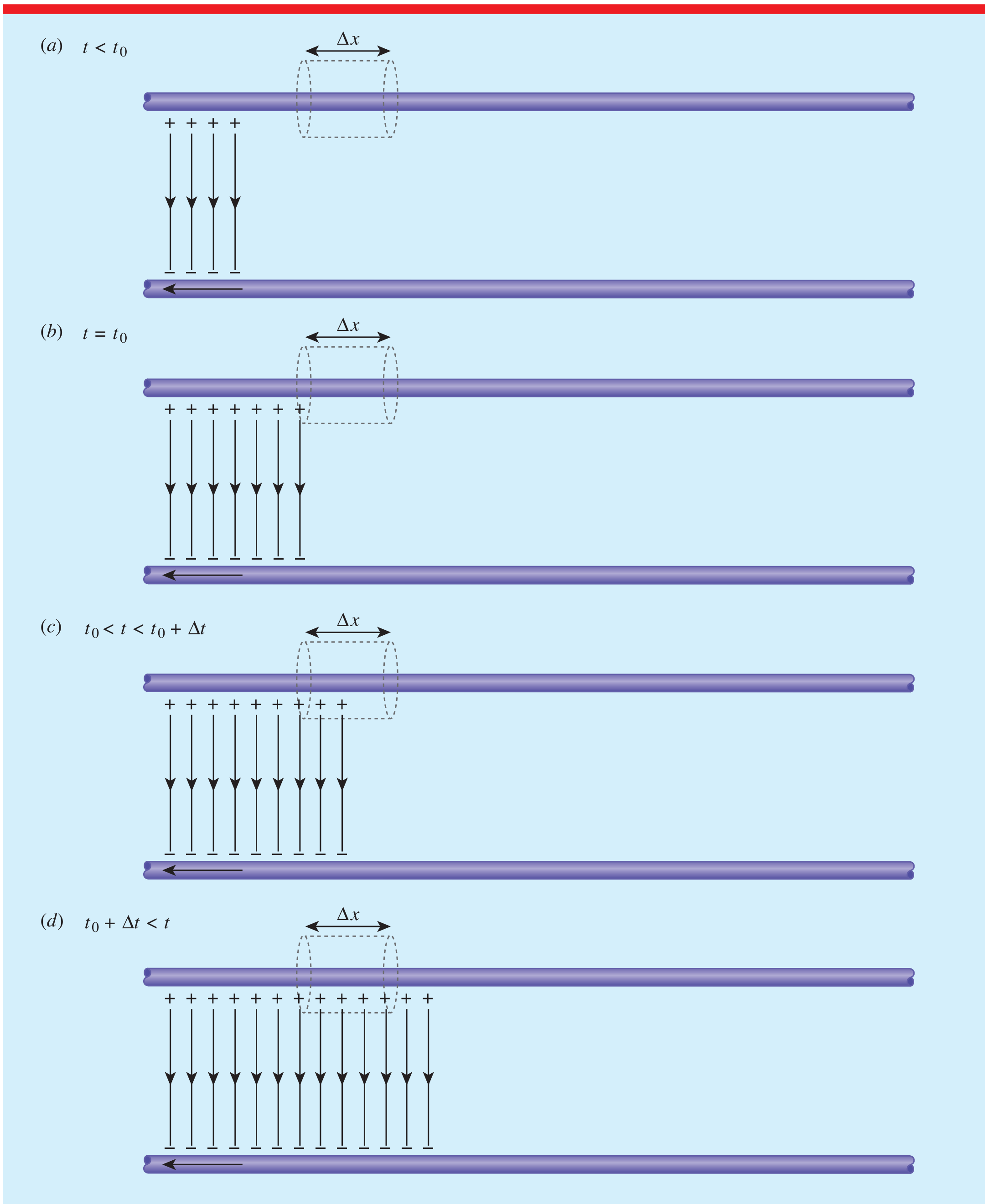

Figure 2. A wave travelling at c along a transmission line and sample volume of length Δx across the upper conductor (a). The wave arrives at the sample volume in (b); it travels through it in (c) and finally exits from the volume in (d).

of free electrons in the metal (for copper it is 8.4830×10^{28} m^{-3}).

It is very important to note that during the time interval Δt this current enters the wire length Δx through its left side, but it does not exit from the other side, as the wave has not yet arrived there. This incoming current lasts for a time interval Δt and produces in the wire length Δx

an imbalance of charge ΔQ given by

$$\Delta Q = I\Delta t = I\frac{\Delta x}{c}. \qquad (2)$$

After Δt has elapsed, the current starts to flow out of our sampling volume and the charges entering from the left are balanced by those escaping towards the right.

Let us consider the wire after the wave has passed (figure 2(d)). The electric field **E** is of course directed normally to the wire surface, and if the wire is very thin and the two wires are not too close the charge can be considered uniform on the wire surface, and hence also the amplitude of **E** is uniform, so Gauss's law can be applied to the cylindrical sampling volume tightly enveloping the wire in a straightforward manner,

$$\frac{\Delta Q}{\varepsilon_0} = (2\pi a\Delta x)E, \qquad (3)$$

with ε_0 the dielectric permittivity of vacuum (8.854×10^{-12} F m^{-1}) and E the amplitude of the electric field **E**. The speed v the electrons are required to move with is obtained by combining the three above equations and is

$$v = \frac{2c\varepsilon_0 E}{qNa}. \qquad (4)$$

The notable point of this result is that the necessary speed decreases with the number of electrons per volume unit N. Therefore, a great number of slow electrons are able to generate enough unbalanced charge to follow an electromagnetic wave travelling at much higher speed.

In order to check the plausibility of (4) in a realistic worst case scenario, we can consider, for example, a copper wire of 0.5 mm radius and an electric field at the limit of dielectric breakdown of air (about 3×10^6 V m^{-1}). In this case (4) gives for the electrons a speed of $v = 2.3$ mm s^{-1}, which is a very reasonable speed.

A deeper insight

The explanation could even end at this point. Nevertheless, in the most advanced classes or as an answer to a possible question, the teacher could cite the *skin effect* and the fact that at the highest frequencies the line acts as a radiating antenna.

Up to this point, the current has been considered constant in the wire section, but in reality the current flow tends to be bound to the portion of the conductor closer to the surface. While the current vanishes exponentially with the depth in the conductor, it is common practice to define a *skin depth* δ and consider the current uniform from the surface to a depth δ and null beyond this depth. The skin depth is

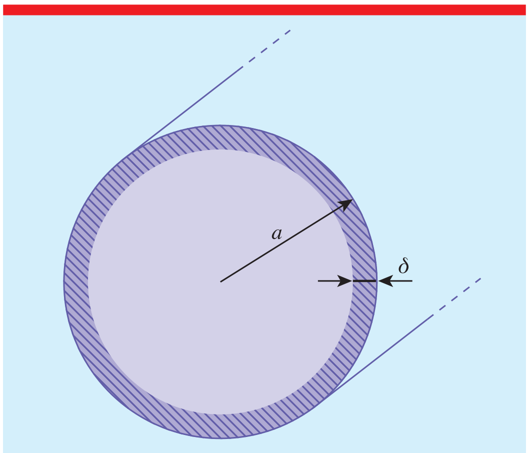

Figure 3. Cross-section of a cylindrical wire at high frequency. The part of the cross-section where current density is concentrated is hatched.

$$\delta = \sqrt{\frac{1}{\pi f \sigma \mu}}, \qquad (5)$$

with f the frequency, σ the conductivity and μ the magnetic permeability of the wire material.

At very high frequencies δ can become much smaller than the wire radius, and hence the current actually crosses a small outer ring of the cross-section of the wire (figure 3).

Therefore, at high frequency (1) should be written more correctly as

$$I = 2\pi a\delta vqN \qquad (6)$$

and, therefore, following the steps above, the drift speed of electrons is given by

$$v = \frac{2c\varepsilon_0 E}{qN\delta}. \qquad (7)$$

For copper, $\sigma = 5.96 \times 10^7$ S m^{-1} and $\mu = 4\pi \times 10^{-7}$ H m^{-1}; therefore, $\delta = 0.1156/\sqrt{f}$ and

$$v = 9.01 \times 10^{-6}\sqrt{f}. \qquad (8)$$

The effective speed depends on the wave frequency: for $f = 10$ GHz equation (8) leads to $v = 0.9$ m s^{-1}. On the other hand, the frequency has a physical limit in a transmission line: when half a wavelength is larger then the distance between the conductors, the transmission line radiates and does not guide the field. As an example, two wires at a distance of 10 mm are no longer a guide but a radiating antenna when f is larger than 15 GHz.

Conclusion

Catt's anomaly is just an exercise that can be solved with the conceptual tools of basic electromagnetism; nevertheless, it is intriguing and able to stimulate reflection on the deep relationship between field and current in a transmission line.

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Massimiliano Pieraccini graduated in physics in 1994 ('Nello Carrara' degree prize) at the University of Florence, Italy, and received his PhD in non-destructive testing in 1998. In 1995, he joined the Department of Electronics and Telecommunications of the University of Florence, Italy, where he is an associate professor. He has been principal investigator and manager of several research projects funded by the European Community, the Italian Research Ministry and private companies. He is the author of ~100 scientific articles, including more than 60 in peer-reviewed international journals.

Stefano Selleri obtained his first degree (*cum laude*) in Electronic Engineering and his PhD in Computer Science and Telecommunications from the University of Florence in 1992 and 1997, respectively. He has been a visiting scholar/professor or researcher at a number of international institutions. He is currently an assistant professor at the University of Florence, where he conducts research on numerical modelling of microwave devices and circuits, with particular attention to numerical optimization.