CWPP
28/8/12

The Catt Question

Dear Mr Catt

I'm concerned — indeed appalled — that your question has waited for so long for an answer, and apparently caused so much controversy. It seems to me a very straightforward question which can be answered with reference to equations whose validity is unquestioned, and thus in a way which should command universal assent.

1. The Fundamental Principle.

Conservation of charge is a fundamental cornerstone of electromagnetism - indeed it was this principle that led Maxwell to introduce a new term into the equations that now bear his name. It is expressed by a local conservation law, which can be given in three forms depending on the dimensionality of the current structure:

One dimension

We have a line current $I(x,t)$ and line charge $q(x,t)$. Charge conservation requires

$$\frac{\partial q}{\partial t} + \frac{\partial I}{\partial x} = 0 \qquad (1)$$

Two Dimensions

We have a surface charge σ and current sheet K. If the surface is locally plane and we can use Cartesian coordinates the charge conservation requires

$$\frac{\partial \sigma}{\partial t} + \frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} = 0. \qquad (2)$$

Three Dimensions

This is the case normally treated by textbooks, where we have charge density $\rho(\underline{r}, t)$ and current density $\underline{J}(\underline{r}, t)$ which satisfy

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0. \qquad (3)$$

All of these equations tell the same story:

> Changes in charge distribution
> result from local convergence or
> divergence of current.

Thus in any fully-specified electromagnetic description of a ~~fully~~ particular physical situation the origin of any charges can be diagnosed by looking at the currents — there can never be any doubt as to where charges come from.

Thus in order to answer the Catt Question we have to give an electromagnetic description of the two-wire transmission line:

The Straight Two-Wire TEM Line

Signals are propagated along a transmission line formed of two perfect conductors lying parallel to the z axis. (The restriction to perfect conductors is obviously a simplification, but it seems to be one implied by the formulation of the question, since it envisages signals propagating unchanged along the line).

We define the transverse gradient operator $\nabla_t = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ 0 \end{pmatrix}$ and a unit vector in the z direction $\hat{\underline{k}}$.

The fields in the vacuum space between the conductors are then given by

$$\underline{E} = -\nabla_t \phi \; V(t - z/c) \qquad \underline{B} = -\frac{1}{c}\hat{k} \times \nabla_t \phi \; V(t - z/c)$$

where $V(\tau)$ is the voltage signal being propagated along the line at the speed of light c, and ϕ is the unique solution of the two-dimensional Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

with boundary conditions $\phi = 0$ on the surface of the return conductor, and $\phi = 1$ on the surface of the signal conductor. The fields inside the conductors are zero. This discontinuity requires surface charges and currents: (n denotes the outward normal, and t denotes transverse component)

$$\sigma = \epsilon_0 E_n = -\epsilon_0 \frac{\partial \phi}{\partial x_n} V(t - z/c)$$

$$K_z = \frac{1}{\mu_0} B_t = \frac{-1}{\mu_0 c} \frac{\partial \phi}{\partial x_n} V(t - z/c)$$

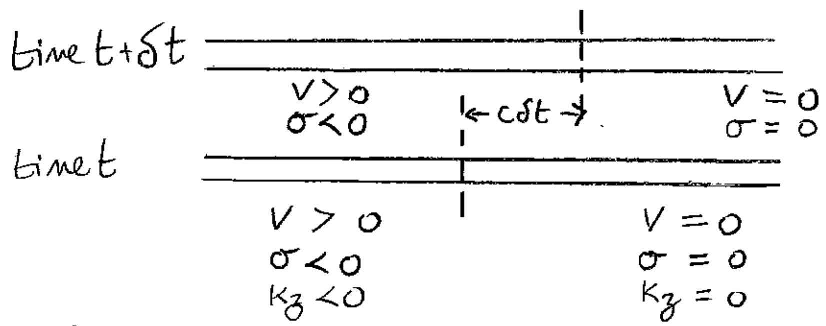
Which satisfy, as they must, (remembering $\frac{1}{\mu_0 c^2} = \epsilon_0$)

$$\frac{\partial \sigma}{\partial t} + \frac{\partial K_z}{\partial z} = 0 \quad \text{which is equation (2)}$$

$\frac{\partial \phi}{\partial x_n}$ is positive on the return line, negative on the signal line, giving, for an upward transition in V, positive charge on the signal line, negative on the return. This charge is delivered by the currents, which only flow in the z direction, so any answers involving charge coming from the interior of the conductor are, within the framework of this model, <u>categorically false</u>.

If the charge is delivered by the currents flowing along the line, does this imply they travel at the speed of light? <u>Definitely not</u>! The only thing that travels at c is the TEM wave that initiates the currents and charges. The charge carriers travel at the drift velocity v.

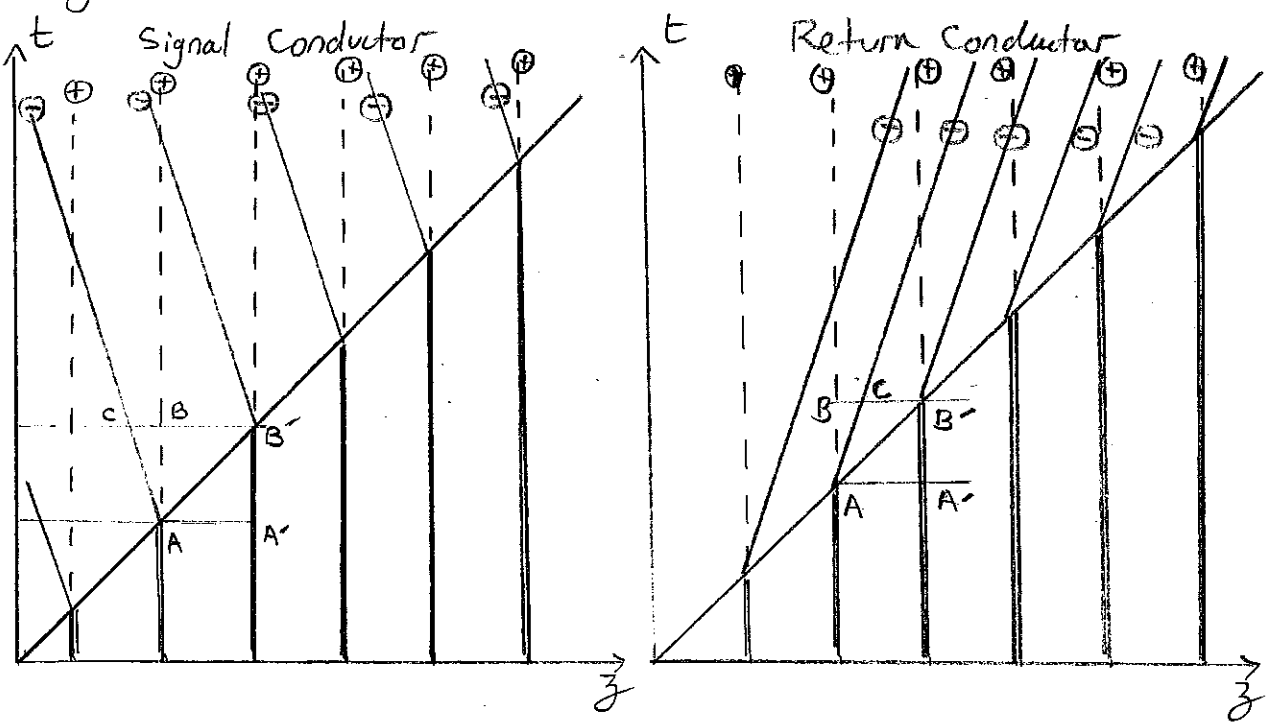
The sketch shows the return line at two times t, $t+\delta t$, between which the wave advances a distance $c\delta t$.



The advancing wave causes currents to flow in a previously uncharged section of line. This spatially inhomogeneous current, which on the return line illustrated is <u>divergent</u> ($\partial k_z/\partial z < 0$), necessarily creates negative charge. Thus the charge on the return conductor is created by separation of positive and negative charges already present on the surface of the conductor. This is the answer to the Catt Question.

This is as far as electromagnetism, as a macroscopic continuum theory, can take us. (Although it is worth remarking the generality of the answer – it has not been necessary to specify the geometry of the conductors – for example whether they are co-axial or parallel – nor the corresponding solution $\varphi(x,y)$, nor the precise form of the signal being transmitted $V(\tau)$.) However if add <u>one microscopic fact</u> – that the conductor consists of fixed positive charges and mobile negative charges – then we can illustrate the mechanism explicitly.

The two space-time diagrams below show a line of fixed and mobile charges as the TEM wave transition from $V = 0$ to $V = V_0$ passes, These charges are on the surface of the signal and return conductors.



As the wave passes each atom the positive charge remains stationary - for example the lines AB - while the electrons begin to drift - ~~from~~ example from A to C on the diagram. The distance BC is thus $v\,\delta t$. If, as shown, δt is the time for the wave to advance to the next atom, with atomic spacing a, then $\delta t = a/c$.

Thus while the positive charges are still spaced by a, the spacing of the electrons is now:

Signal Conductor: $a(1 + v/c)$

Return Conductor $a(1 - v'/c)$

There is thus a net charge per unit length:

Signal: $q = e/a - e/a(1+v/c) = \dfrac{ev/c}{1+v/c}$

Return: $q' = e/a - \dfrac{e}{a(1-v'/c)} = -\dfrac{e v'/c}{1-v'/c}$

where v and v' are the drift velocities on the two lines. Similarly the net current on the two lines (from this particular line of charges) is

$$i = 0 + \frac{(-e)(-v)}{a(1+v/c)} = \frac{ev}{a(1+v/c)} = cq$$

for the signal and

$$i = 0 + \frac{(-e)(v')}{a(1-v'/c)} = \frac{-ev'}{a(1-v'/c)} = cq'$$

on the return line. The drift velocity must be such that at each point on the surface of each conductor the surface current has the correct value determined by the equations above, but this crude model illustrates how the creation of the current as the wave passes necessarily leads to a charge density in the correct ratio.

The phenomenon of decrease in density of a line of objects as ~~f~~ a 'wave of starting' propagates in the <u>opposite</u> direction to the resulting motion, as on the signal conductor, is very familiar. Consider a line of stationary traffic at a traffic light on red, spaced by 5m, facing the $-z$ direction. As the light turns green the cars rapidly accelerate to $15\,ms^{-1}$, and the wave of starting propagates at about $2\,ms^{-1}$, giving a spacing in the resulting traffic flow of

$$a(1+v/c) = 5(1+{}^{15}/_{2}) = 42\,m.$$

(The numbers are only roughly realistic!)

This is a necessary consequence of the divergent traffic flow, corresponding to the divergent electron flow (convergent current flow) of the signal line.

The increase in density resulting from starting a queue into motion from the rear instead of the front, is fortunately less familiar, but is the process producing the charge on the return line.