ARTHUR F. KIP
*Professor of Physics, University of California, Berkeley*

# *Fundamentals of Electricity and Magnetism*

McGRAW-HILL BOOK COMPANY, INC.
*New York San Francisco Toronto London 1962*

We have now written three of the four fundamental relationships of Maxwell. In the next section we see the way in which Maxwell was led to the fourth and final relation, involving the new concept of *displacement current.*

### *12.3 Displacement Current*

In the process of unifying electromagnetic theory, Maxwell discovered the final relationship that leads to the possibility of electromagnetic waves. This relationship is the converse of Faraday's law, as given in Eq. (12.1). Maxwell showed that just as a varying magnetic field gives rise to an electric field, a varying electric field gives rise to a magnetic field. The argument shown here is similar to the one given by Maxwell.

We begin by examining the generalized form of Ampère's law for the production of a magnetic field by a current:

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i \qquad \text{or} \qquad \oint \mathbf{H}\cdot d\mathbf{l} = i \tag{6.14}$$

An equivalent form is

$$\oint \mathbf{H}\cdot d\mathbf{l} = \int \mathbf{j}\cdot d\mathbf{S} \tag{12.4}$$

Here the line integral is taken around any closed path, and the surface integral is taken over *any* surface bounded by the path of the line integral. The surface integral of the current density $\mathbf{j}$ is the total current threading the path of the line integral. When we choose the path so that $\mathbf{H}$ is constant and parallel to the path, $\mathbf{H}$ can be evaluated, as we have seen earlier. Figure 12.1 illustrates the idea that *any* surface bounded by the path of the line integral gives the same result. In the case of a wire carrying the current, we know this is true since the surface integral is always equal to $i$

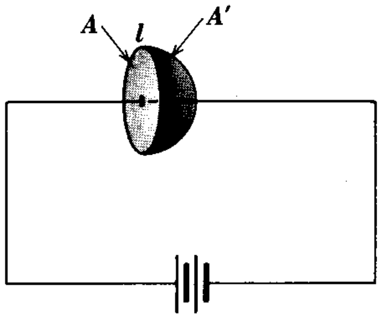


***Fig. 12.1** Two areas A and A′ bounded by the same closed line l.*

as long as the wire threads the path. The figure shows one plane surface $A$ and one hemispherical surface $A'$, both bounded by the same line $l$.

We now examine the situation when the battery is replaced by an a-c source and a capacitor is included in the circuit, as shown in Fig. 12.2. An a-c current passes through a capacitor, as we have

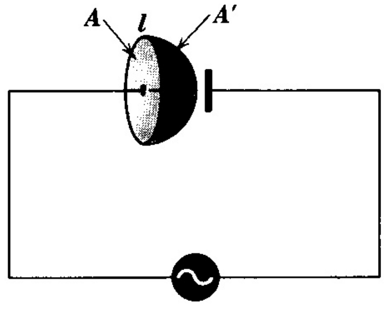


***Fig. 12.2** The integral $\oint \mathbf{H} \cdot d\mathbf{l}$ over the area $A$ is $i$; it is zero over $A'$ unless displacement current is taken into account.*

seen earlier, though no actual charge is transferred between the plates. Maxwell's reasoning involved taking the two areas $A$ and $A'$, bounded by the same path $l$. He pointed out that according to Eq. (12.4), when the surface integral is taken over $A$, the integral $\oint \mathbf{H} \cdot d\mathbf{l} = i$, but if it is taken over $A'$, the integral equals 0. This contradictory situation can be remedied only if we postulate, with Maxwell, an additional term in Eq. (12.4) in which the changing electric field occurring in the capacitor takes the place of the real current as a producer of the magnetic field. We thus arrive at Maxwell's displacement current and remove the double meaning of the mathematics of Eq. (12.4) in the a-c case.

If we use a simple parallel-plate capacitor, it is easy to arrive at the required relationship. Since the current is the rate at which charge accumulates on the capacitor plates (since charge is conserved), we can write

$$i = \int \mathbf{j} \cdot d\mathbf{S} \tag{12.5}$$

But $i = dq/dt = A\, d\sigma/dt$, or $j = d\sigma/dt$. We have already seen that between the plates of a parallel-plate capacitor, $D = \sigma$, so $d\sigma/dt$ can be replaced by $dD/dt$. Then Eq. (12.4) becomes

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \frac{d\mathbf{D}}{dt} \cdot d\mathbf{S} \tag{12.6}$$

in the region between the capacitor plates. That is, the displacement current $dD/dt$ acts like a current density $j$ in that it produces a region of magnetic field. The complete expression involving both terms, the magnetic effect of the real current in the wire and the magnetic effect of the change in $D$, can be written as

$$\oint \mathbf{H}\cdot d\mathbf{l} = \int \mathbf{j}\cdot d\mathbf{S} + \int \frac{d\mathbf{D}}{dt}\cdot d\mathbf{S} \tag{12.7}$$

For most situations involving conducting bodies, the new term involving the displacement current is trivial compared with the current term at low frequencies. In a vacuum it is the only term.

Since we show below that this new concept of magnetic field generation by displacement current is necessary to account for electromagnetic waves, we may consider the proved existence of such waves as the final proof of the validity of Maxwell's reasoning.

The development of the idea that light is an electromagnetic radiation is one of the most dramatic in the history of physics. We shall quote below from the introduction to the paper [1] in which Maxwell announced the electromagnetic theory of light. In the introduction, Maxwell summarizes the results of his mathematical investigations, describing eight relationships, which he calls the general equations of the electromagnetic field. These are essentially equivalent to the four relationships now called Maxwell's equations. Maxwell then goes on to discuss some of the implications of his results in the following paragraphs:

> The general equations are next applied to the case of a magnetic *disturbance propagated through a nonconducting field, and it is shown* that the only disturbances which can be so propagated are those which are transverse to the direction of propagation, and that the velocity of propagation is the velocity $v$, found from experiments such as those of Weber, which expresses the number of electrostatic units of electricity which are contained in one electromagnetic unit.
>
> This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws. If so, the agreement between the elasticity of the medium as calculated from the rapid alternations of luminous

[1] James Clerk Maxwell, A Dynamical Theory of the Electromagnetic Field, *Phil. Trans. Roy. Soc. London*, **155**:459 (1865).

vibrations, and as found by the slow processes of electrical experiments, shows how perfect and regular the elastic properties of the medium must be when not encumbered with any matter denser than air. If the same character of the elasticity is retained in dense transparent bodies, it appears that the square of the index of refraction is equal to the product of the specific dielectric capacity and the specific magnetic capacity. Conducting media are shown to absorb such radiations rapidly, and therefore to be generally opaque.

The conception of the propagation of transverse magnetic disturbances to the exclusion of normal ones is distinctly set forth by Professor Faraday in his "Thoughts on Ray Vibrations." The electromagnetic theory of light, as proposed by him, is the same in substance as that which I have begun to develop in this paper, except that in 1846 there were no data to calculate the velocity of propagation.

## *12.4 The Physics and Mathematics of Waves*

Our next task is to see how Maxwell's general formulation of electromagnetism leads to self-propagating waves. Before doing this, however, it is useful to review the simple theory of traveling waves. We discuss the simple situation of wave propagation on a stretched string.